Note on Lie Groups, Lie Algebras and Representations:
An Elementary Introduction by Brain C. Hall

Chapter 1. Matrix Lie Groups

Def. Let $\{A_m\}$ be a sequence of matrice in $M_n(\mathbb{C})$, We say that A_m converges to a matrix A of each entry $(A_m)_{ij} \xrightarrow{m\to\infty} A_{ij}$.

Def. A matrix Lie Group is a subgroup G of $GL(n;\mathbb{C})$ with the following property: If $\{A_m\} \subseteq G$ and $A_m \to A$, then $A \in G$ or A invertible.

Rmk. matrix Lie group $\Leftrightarrow$ closed subgroup of $GL(n;\mathbb{C})$

Example.

- $GL(n;\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A \text{ invertible}\}$
  $GL(n;\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ invertible}\}$ General Linear Groups
- $SL(n;\mathbb{C}) = \{A \in GL(n;\mathbb{C}) : |A| = 1\}$
  $SL(n;\mathbb{R}) = \{A \in GL(n;\mathbb{R}) : |A| = 1\}$ Special Linear Groups
- $U(n) = \{A \in GL(n;\mathbb{C}) : A^* A = I\}$ $(A^* = \bar{A}^T)$
  $= \{A \in GL(n;\mathbb{C}) : \langle Ax, Ay\rangle = \langle x, y\rangle, \forall x, y \in \mathbb{C}^n\}$ $(\langle x, y\rangle = \sum_{i=1}^{n} \bar{x}_i y_i)$ Unitary Groups
- $SU(n) = \{A \in U(n) : |A| = 1\}$ Special Unitary Groups

Rmk. $\forall A \in U(n)$, $|\det A| = 1$

- $O(n) = \{A \in GL(n;\mathbb{R}) : A^T A = I\}$
  $= \{A \in GL(n;\mathbb{R}) : (Ax, Ay) = (x, y), \forall x, y \in \mathbb{R}^n\}$ $((x,y) = \sum_{i=1}^{n} x_i y_i)$ Orthogonal Groups

  $O(n;\mathbb{C}) = \{A \in GL(n;\mathbb{C}) : A^T A = I\} = \{A \in GL(n;\mathbb{C}) : (Ax, Ay) = (x, y) \ \forall x, y \in \mathbb{C}^n\}$
- $SO(n) = \{A \in O(n) : |A| = 1\}$
  $SO(n;\mathbb{C}) = \{A \in O(n;\mathbb{C}) : |A| = 1\}$ Special Orthogonal Groups
- Def. $[\cdot,\cdot]_{n,k} : \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \to \mathbb{R} : [x, y]_{n,k} \mapsto x^T g y$, $g = \begin{pmatrix} I_n & \\ & -I_k \end{pmatrix}$
- $O(n;k) = \{A \in GL(n+k, \mathbb{R}) : A^T g A = g\}$
  $= \{A \in GL(n+k, \mathbb{R}) : [Ax, Ay] = [x, y], \forall x, y \in \mathbb{R}^n\}$ Generalized Orthogonal Groups

In Particular,
$SO(n;k) = \{A \in O(n;k) : |A| = 1\}$
$O(3,1)$ is the Lorentz Group.

- $\omega(x,y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$, $x, y \in \mathbb{R}^{2n}$

$= x^T \Omega y$, where $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

- $Sp(n;\mathbb{R}) = \{A \in GL(n;\mathbb{R}) : A^T \Omega A = \Omega\}$ Real Symplectic Groups

$= \{A \in GL(n;\mathbb{R}) : A \text{ preserves } \omega\}$

Rmk. $\forall A \in Sp(n;\mathbb{R})$, $|A| = 1$.

- $Sp(n;\mathbb{C}) = \{A \in GL(n;\mathbb{C}) : \Omega = A^T \Omega A\}$ Complex Symplectic Groups

- $Sp(n) = Sp(n;\mathbb{C}) \cap U(2n)$ Compact Symplectic Groups

Theorem. $U \in Sp(n)$ if and only if there exists an orthonormal basis $u_1, \cdots, u_n$, $v_1, \cdots, v_n \in \mathbb{C}^{2n}$, s.t.

1) $J u_i = v_i$ 2) $U u_j = e^{i\theta_j} u_j$, $U v_j = e^{-i\theta_j} v_j$ for $\theta_1, \cdots, \theta_n \in \mathbb{R}$

3) $\omega(u_j, u_k) = \omega(v_j, v_k) = 0$, $\omega(u_j, v_k) = \delta_{jk}$

where J is a conjugate-linear map : $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by $J(\alpha, \beta) = (-\bar{\beta}, \bar{\alpha})$ $\forall \alpha, \beta \in \mathbb{C}^n$

Topological Properties

Def. A matrix Lie group G is compact if

1) $\forall \{A_m\} \subseteq G$ and $A_m \to A$, then $A \in G$

2) $\exists$ constant C , s.t. $|A_{jk}| \le C$ $\forall 1 \le j, k \le n$.

Def. G is connected iff $\forall A, B \in G$, $\exists$ continuous path $A(t)$, s.t. $A(a) = A$, $A(b) = B$. Then the identity component $= \{\text{connected with } I\}$

- identity component is a normal subgroup of G

- $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$, $SU(n)$ are connected

Def. G is simply connected iff every loop on G can be shrunk continuously to a point in G

- $SU(n)$ is simply connected.

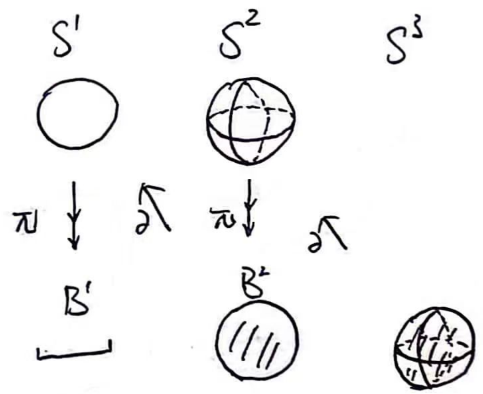
$\pi : D^3 \twoheadrightarrow \mathbb{RP}^3$

$\Phi : B^3 \twoheadrightarrow SO(3)$

$u \mapsto R_{\hat{u}, \pi\|u\|}$

Theorem. $SO(3) \cong \mathbb{RP}^3$ as a topology

Theorem. $SU(2) \cong S^3$



Def. A Lie group is a smooth manifold and also a group, such that the group product and the inverse map are smooth.

Rmk. Not every Lie group is isomorphic to a matrix Lie group.

Chapter 2. The Matrix Exponential

Def. $e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$, $\forall X \in Mat_n(\mathbb{C})$ & $\|X\| = \left(\sum_{j,k=1}^{n} |X_{jk}|^2\right)^{\frac{1}{2}}$

- e^X converges for all $X \in Mat_n(\mathbb{C})$ and e^X is a continuous.
- $C \in GL(n;\mathbb{C}) \Rightarrow e^{CXC^{-1}} = Ce^XC^{-1}$

Def. $\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$ whenever it converges.

- Generally, $\|A-I\|<1 \Rightarrow \log A$ converges. However, the converse is false, e.g. $A-I$ is nilpotent.

- For $\|A-I\|<1$, $e^{\log A} = A$
- For $\|X\| < \log 2$, $\|e^X - I\| < 1$ and $\log e^X = X$.

  Pf. $\|e^X - I\| \le \sum_{k=1}^{\infty} \frac{\|X\|^k}{k!} = e^{\|X\|} - 1 < 1$

  $$X = CDC^{-1},\ \log e^X = C \log e^D C^{-1} = CDC^{-1} = X$$

- $\exists C \in \mathbb{R}$, s.t. $\forall B$ with $\|B\| < \frac{1}{2}$, we have $\|\log(I+B) - B\| \le C\|B\|^2$ (or $\log(I+B) = B + O(\|B\|^2)$)

- $\forall$ invertible matrix can be expressed by e^X for some $X \in M_n(\mathbb{C})$
- $\det(e^X) = e^{\operatorname{tr}(X)}$

Theorem. (Lie Product Formular) $e^{X+Y} = \lim_{m\to\infty} (e^{\frac{X}{m}} e^{\frac{Y}{m}})^m$, $X, Y \in M_n(\mathbb{C})$

pf. $(e^{\frac{X}{m}} e^{\frac{Y}{m}})^m = \left(I + \frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2})\right)^m$

$\Rightarrow \log(e^{\frac{X}{m}} e^{\frac{Y}{m}})^m = m \log\left(I + \frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2})\right)$

$= X + Y + O(\frac{1}{m}) \longrightarrow X+Y$

$\Rightarrow \lim_{m\to\infty} (e^{\frac{X}{m}} e^{\frac{Y}{m}})^m = \exp(X+Y)$

Def. A function $A: \mathbb{R} \to GL(n;\mathbb{C})$ is called a one-parameter subgroup of $GL(n;\mathbb{C})$ if. $\forall a, b \in \mathbb{R}$

1) A is continuous, (2) $A(0) = I$ (3) $A(a)A(b) = A(a+b)$

- Actually, this subgroup $\Leftrightarrow e^{tX} = A(t)$, $t \in \mathbb{R}$

- The Polar Decomposition

1) $\forall A \in GL(n;\mathbb{C})$, $A = UP$, where U is unitary, P is self-adjoint and positive.

$\downarrow \Leftarrow$ P self-adjoint & positive $\Leftrightarrow P = e^X$ (unique)

$A = Ue^X$, where U is unitary, X self-adjoint.

and X depend continuously on A

2) $\forall A \in GL(n;\mathbb{R})$, $A = Re^X$, where $R \in O(n)$, X real and symmetric

3) $\forall A \in SL(n;\mathbb{C})$, $A = Ue^{X}$, $U \in SU(n)$ & X self-adjoint and trace zero.

4) $\forall A \in SL(n;\mathbb{R})$, $A = Re^{X}$, $R \in SO(n)$ & X real, symmetric and trace zero.

Chapter 3 Lie Algebras (Always assume G is a matrix Lie group with the Lie alg $\mathfrak{g}$)

I won't go into too much detail for defs here, cause we're already familiar with them.

Def. G is a matrix group. The Lie algebra of G, denoted $\mathfrak{g}$, is the set of all matrices such that $e^{tX} \in G$ for $t \in \mathbb{R}$.

- If $X \in \mathfrak{g}$, then $e^{X} \in G_0$ (the identity component of G)
- If $X \in \mathfrak{g}$, $AXA^{-1} \in \mathfrak{g}$ for all $A \in G$ and $sX \in \mathfrak{g}$ $\forall s \in \mathbb{R}$
- $\mathfrak{g}$ is a real Lie algebra with the commutator.

If $\mathfrak{g}$ is a complex subspace of $M_n(\mathbb{C})$, then $\mathfrak{g}$ is complex.

- If G is commutative, then $\mathfrak{g}$ is commutative.

Ex.

Matrix Lie Group	Its Lie Algebra
$GL(n;\mathbb{C})$ $(GL(n;\mathbb{R}))$	$\mathfrak{gl}_n(\mathbb{C})$ $(\mathfrak{gl}_n(\mathbb{R}))$
$SL(n;\mathbb{C})$ $(SL(n;\mathbb{R}))$	$\mathfrak{sl}_n(\mathbb{C})$ $(\mathfrak{sl}_n(\mathbb{R}))$
$U(n)$ $(SU(n))$	$\mathfrak{u}(n)$ $(\mathfrak{su}(n))$ $X^* = -X$ & $\mathrm{tr}(X) = 0$; $X^* = -X$

Matrix Lie Group	Its Lie Algebra
$O(n)$ $(SO(n))$	$\mathfrak{so}_n(\mathbb{R}) \to X^T = -X$
$O(n;k)$ $(SO(n;k))$	$\mathfrak{so}_n(k) \to gX^Tg = -X$
$Sp(n;\mathbb{C})$ $(Sp(n;\mathbb{R}))$	$\mathfrak{sp}(n;\mathbb{C})$ $(\mathfrak{sp}(n;\mathbb{R}))$ $\to \Omega X^T \Omega = -X$
$Sp(n)$	$\mathfrak{sp}(n) \to \Omega X^T \Omega = -X$ & $X^* = -X$

Thm. Let G and H be matrix Lie groups, with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ respectively. Suppose $\Phi: G \to H$ is a Lie group homo, There exist a Lie alg homo $\phi: \mathfrak{g} \to \mathfrak{h}: X \mapsto \frac{d}{dt}\Phi(e^{tX})\big|_{t=0}$.

- $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$
- $\Phi: H \to K$, $\Psi: G \to H$ (Lie grp homos) and ϕ, ψ respectively. The Lie alg homo λ of the composition $\Phi \circ \Psi$ is $\phi \circ \psi$.
- $\ker \Phi$ is a closed, normal subgroup of G and satisfies $\mathrm{Lie}(\ker \Phi) = \ker \phi$.
  $\to$ the corresponding Lie alg.
- $\mathrm{Ad}: A \to (\mathrm{Ad}_A : X \mapsto AXA^{-1} : \mathfrak{g} \to \mathfrak{g})$ is a Lie group homo ($A \in G$) G to $GL(\mathfrak{g})$. and the corresponding Lie alg homo is ad.
- $\forall X \in M_n(\mathbb{C})$, $e^{X} Y e^{-X} = \mathrm{Ad}_{e^X} Y = e^{\mathrm{ad}_X}(Y)$

• universal property of the complexification of a real Lie alg:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\Phi_{\mathbb{R}}"] \arrow[d, hook] \arrow[dr, phantom, "G"] & \mathfrak{h} \\ \mathfrak{g}_{\mathbb{C}} \arrow[ur, dashed, "\Phi_{\mathbb{C}}"'] & \end{tikzcd}$$

Def. The exponential map for G is the map $\exp: \mathfrak{g} \to G$.

• $\exp: \mathfrak{gl}_n(\mathbb{C}) \to GL(n;\mathbb{C})$ is bijective, but in general, it's not true. (e.g. there does not exist $X \in \mathfrak{sl}_2(\mathbb{C})$ s.t. $e^X = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} \in SL_2(\mathbb{C})$)

Theorem. Suppose $G \subset GL(n;\mathbb{C})$ with Lie alg $\mathfrak{g}$. Then there exist $\varepsilon \in (0, \log 2)$ such that for all $A \in V_\varepsilon = \exp \mathcal{U}_\varepsilon = \exp(\{X \in M_n(\mathbb{C}) : \|X\| < \varepsilon\})$, A is in G if and only if $\log A$ is in $\mathfrak{g}$.

Pf. Let D denote the orthogonal component of $\mathfrak{g}$ with respect to the inner product on $\mathbb{R}^{2n^2} \cong M_n(\mathbb{C})$.

Consider the map $\Phi: M_n(\mathbb{C}) \to M_n(\mathbb{C}): Z \to e^X e^Y$, where $Z = X + Y$ with $X \in \mathfrak{g}$, $Y \in D$. Then $\frac{d}{dt}\Phi(tX, 0)\big|_{t=0} = X$, and $\frac{d}{dt}\Phi(0, tY)\big|_{t=0} = Y$. So the derivate of Φ at the point 0 is identity: $\mathbb{R}^{2n^2} \to \mathbb{R}^{2n^2}$. Since it is invertible, the inverse function theorem says that Φ has a continuous local inverse.

If for any $\varepsilon \in (0, \log 2)$, $\exists A \in V_\varepsilon \cap G$ s.t. $\log A \notin \mathfrak{g}$, then take $A_m \in V_\varepsilon \cap G$ and $A_m \xrightarrow{m \to \infty} I$. Using the inverse function theorem

$A_m = e^{X_m} e^{Y_m}$, $X_m \in \mathfrak{g}$ and $Y_m \in D$, both converge to zero and $Y_m \neq 0$.

So $e^{Y_m} = e^{-X_m} A_m \in G$, $\forall m \in \mathbb{Z}_+$. Now it suffices to show $Y_m \in \mathfrak{g}$, which is a conflictiion.

$\frac{Y_m}{\|Y_m\|}$'s are in the unit sphere in D. Choose a subsequence of Y_m's s.t. $\frac{Y_m}{\|Y_m\|} \to Y \in D$ & $\|Y\| = 1$. However, for any $t \in \mathbb{R}$, $\exists k_m \in \mathbb{Z}$ s.t. $k_m \|Y_m\| \to t$, which follows $e^{k_m Y_m} = e^{k_m \|Y_m\| \frac{Y_m}{\|Y_m\|}} \to e^{tY} \in G$, because $e^{k_m Y_m} \in G$. $\Rightarrow Y \in \mathfrak{g}$. #

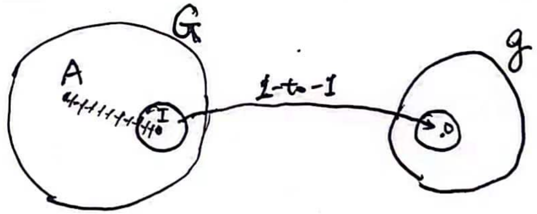
• There exists a neighborhood $\mathcal{U}$ of 0 in $\mathfrak{g}$ and a neighborhood V of I in G such that the exponential map takes $\mathcal{U}$ homeomorphically onto V.

• G is a smooth embedded submanifold of $M_n(\mathbb{C})$ of dimension $\dim_{\mathbb{R}} \mathfrak{g}$ and hence a Lie group.

• $X \in \mathfrak{g}$ if and only if there exists a smooth curve $\gamma(t) \in G$ for $t \in \mathbb{R}$ such that $\gamma(0) = I$ & $\frac{d}{dt}\gamma\big|_{t=0} = X$. Thus, $\mathfrak{g}$ is the tangent space at the identity to G.

• If G connected, $\forall A \in G$, A can be expressed in the form

$$A = e^{X_1} e^{X_2} \cdots e^{X_m}, \text{ where } X_i \in \mathfrak{g},\ i \in \{1, \cdots, m\}.$$



sketch of pf.

1. segment 2. pf $\|A^{-1}(\frac{j-1}{m}) A(\frac{j}{m}) - I\| < \varepsilon$

3. $A(1) = A(1) A^{-1}(\frac{m-1}{m}) \cdots A(\frac{1}{m}) A^{-1}(0) A(0) = \cdots$

- $\Phi_1, \Phi_2 : G \to H$ and G is connected, and ϕ_1 & ϕ_2 are the corresponding Lie alg homos of $\mathfrak{g}$ into $\mathfrak{h}$. Then $\phi_1 = \phi_2 \Leftrightarrow \Phi_1 = \Phi_2$

Pf. $\forall A \in G$, $A = e^{X_1} \cdots e^{X_m}$,

$$\Phi_1(A) = \Phi_1(e^{X_1} \cdots e^{X_m}) = e^{\phi_1(X_1)} \cdots e^{\phi_1(X_m)}$$
$$= e^{\phi_2(X_1)} \cdots e^{\phi_2(X_m)} = \Phi_2(e^{X_1} \cdots e^{X_m}) = \Phi_2(A)$$

- $\forall$ continuous homo between G and H is smooth.
- G commutative $\Rightarrow$ $\mathfrak{g}$ commutative.

  G connected, then G commutative $\Leftarrow$ $\mathfrak{g}$ commutative,
- The identity component G_0 of G is also a matrix Lie group, and the Lie alg $\mathfrak{g}_0 = \mathfrak{g}$.